ON THE PRODUCT OF THE DISTANCES OF A POINT FROM THE VERTICES OF A POLYTOPE

BY

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ABSTRACT

Let $x_1, \ldots, x_m$ be points in the solid unit sphere of E_n and let x belong to the convex hull of $x_1, \ldots, x_m$. Then $\prod_{i=1}^{m} \| x - x_i \| \leqq (1 - \|x\|)(1 + \|x\|)^{m-1}$. This implies that all such products are bounded by $(2/m)^m(m-1)^{m-1}$. Bounds are also given for other normed linear spaces. As an application a bound is obtained for $|p(z_0)|$ where $p(z) = \prod_{i=1}^{m}(z - z_i)$, $|z_i| \leqq 1, i = 1, \ldots, m$, and $p'(z_0) = 0$.

Introduction. In § 1 we consider m, not necessarily distinct, points $x_1, \cdots, x_m$ belonging to the solid unit sphere (unit ball) of the real n-dimensional Euclidean space E_n. Let x belong to the convex hull $H = H(x_1, \cdots, x_m)$ of the points x_i. The main result of this paper (Theorem 2) states that under these conditions

$$\prod_{i=1}^{m} \| x - x_i \| \leqq (1 - \| x \|)\,(1 + \| x \|)^{m-1}.$$

($\| x \|$ is the Euclidean norm of x.) We obtain this bound by a simple, but lengthy, geometric argument, which proceeds by induction on the dimension of H (Theorem 1). An immediate corollary to Theorem 2 states that

$$\prod_{i=1}^{m} \| x - x_i \| \leqq \left(\frac{2}{m}\right)^m (m-1)^{m-1},$$

for all sets $(x, x_1, \cdots, x_m)$ satisfying $\| x_i \| \leqq 1$, $i = 1, \cdots, m$, and $x \in H(x_1, \cdots, x_m)$.

In §2 we show that this corollary can also be deduced from Szegö's maximum principle. This principle asserts that the product $\prod_{i=1}^{m} \| x - x_i \|$ attains, for fixed points x_i, its maximum in any bounded closed region only at the boundary of this region. For our purpose it is convenient to formulate a consequence of this principle for the convex polytope $H(x_1, \cdots, x_m)$ (Theorem 3).

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In §3 we consider the solid unit sphere of an arbitrary normed linear space N. We denote by $h_m(N)$ the supremum of $\prod_{i=1}^{m} \| x - x_i \|$ for 1l $\| x_i \| \leqq 1$, $i = 1, \cdots, m$, and all $x \in H(x_1, \cdots, x_m)$ and obtain bounds for $h_m(N)$. The lower bound is attained in the Euclidean case (Corollary to Theorem 2) and the upper bound is attained for spaces with the supremum norm and for spaces with the L_1 norm. We conclude this paper with a simple application of Theorem 2 to complex polynomials having all their roots in the unit disk.

As the title of this paper may be misleading, we stress that the m (not necessarily distinct) points are in an arbitrary position and that x is always restricted to their convex hull H. We consider the product of all the m distances $\| x - x_i \|$, $i = 1, \cdots, m$, and not only the product of the distances from the vertices of the convex polytope H. ∂C and int C denote the boundary and the interior of the convex set C relative to the flat (linear variety) of smallest dimension containing C. This dimension is denoted by dim C. An edge is a one-dimensional face of the convex polytope H (belonging to ∂H). The author wishes to thank Dr. A. Ginzburg of the Technion, Haifa, for his help in the preparation of this paper.

1. Polytopes in euclidean space.

THEOREM 1. *Let S_n be a solid sphere of E_n and let $x, x_1, \cdots, x_m$, $(m \geqq 2)$, be (not necessarily distinct) points of S_n such that x belongs to the convex hull $H = H(x_1, \cdots, x_m)$ of $x_1, \cdots, x_m$. Then there exists points $x_1', \cdots, x_m'$ in S_n such that x lies on an edge of their convex hull $H' = H(x_1', \cdots, x_m')$ and*

$$\prod_{i=1}^{m} \| x - x_i \| \leqq \prod_{i=1}^{m} \| x - x_i' \|. \tag{1}$$

Proof. The left hand side of (1) vanishes only if x coincides with one of the x_i. In this case we choose $x_2' = x_3' = \cdots = x_m' = x$ and take as x_1' any point of S_n different from x. We shall in the sequel disregard this trivial case and always assume that $x \neq x_i$, $i = 1, \cdots, m$. This and $x \in H(x_1, \cdots, x_m)$ imply $x \in \text{int } S_n$. In the conclusion of the theorem x will be an interior point of an edge of H'.

Let $k = \dim H$ and let P_k be the k-flat carrying H. $S_k = S_n \cap P_k$ is a solid k-sphere containing all the x_i. We shall find points x_i', $i = 1, \cdots, m$, belonging to S_k, and hence also to S_n, satisfying the requirements of the conclusion. The proof will be by induction on the dimension k of H and we disregard E_n and S_n.

We denote the center of S_k by c_k. For $x \in \text{int } S_k$, $x \neq c_k$, we denote the point of S_k farthest away from x by $a_k(x)$ and we denote the point of ∂S_k nearest to x by $b_k(x)$; i.e. $a_k(x)$ and $b_k(x)$ are the endpoints of the diameter through x. $a_k(c_k)$ and $b_k(c_k)$ are the endpoints of an arbitrary diameter of S_k. For the induction it is convenient to prove more than stated in the theorem; we show that H' can always be chosen as a segment or as a triangle. Precisely, we prove the following *strong version* of Theorem 1:

Let $k = \dim H(x_1, \cdots, x_m)$, let S_k be a solid k-sphere such that $x_i \in S_k$, $i = 1, \cdots, m$, and let $x \in H$, $x \neq x_i, i = 1, \cdots, m$. We can choose the m points x_i' of S_k which satisfy (1) *and for which x lies on the segment $x_1'x_2'$ in the following way: $x_1' \neq x_2'$, $x_1' \neq a_k(x)$ and (for $m \geqq 3$) $x_3' = \cdots = x_m' = a_k(x)$.*

We prove the strong version by induction on k. For $k = 1$ we choose $x_1' = b_1(x)$ and $x_2' = x_3' = \cdots = x_m' = a_1(x)$. (1) is obvious and $H' = S_1$, hence $x \in H'$.

For $k = 2$ we have three possibilities. (*a*) $x = c_2$. Set $x_1' = b_2(c_2)$ and $x_2' = x_3' = \cdots = x_m' = a_2(c_2)$.

(b) $x \in \partial H$. x is thus an interior point of the segment x_1x_2. (We avoid sub-subscripts. A subset of p points of the given set will always be denoted by $x_1, \cdots, x_p$.) Set $x_1' = x_1(\neq a_2(x))$, $x_2' = x_2$ and $x_3' = \cdots = x_m' = a_2(x)$.

(c) $x \in \operatorname{int} H$, $x \neq c_2$. Let r be the radius through $x (r = c_2b_2(x))$ and let d be its intersection with ∂H. If d is a vertex of H or, more general, if d is one of the points x_i, then we set $x_1' = d$ and $x_2' = x_3' = \cdots = x_m' = a_2(x)$. If d is not a vertex of H, then it is an interior point of the side (edge) x_1x_2 of H. Let l be the line through x parallel to this segment x_1x_2. Clearly, $c_2 \notin l$. Denote the intersection points of l and ∂S_2 by x_1' and x_2', choosing x_i' on the same side of r as x_i, $i = 1, 2$. (Cf. Figure 1). Then we have for $i = 1, 2$

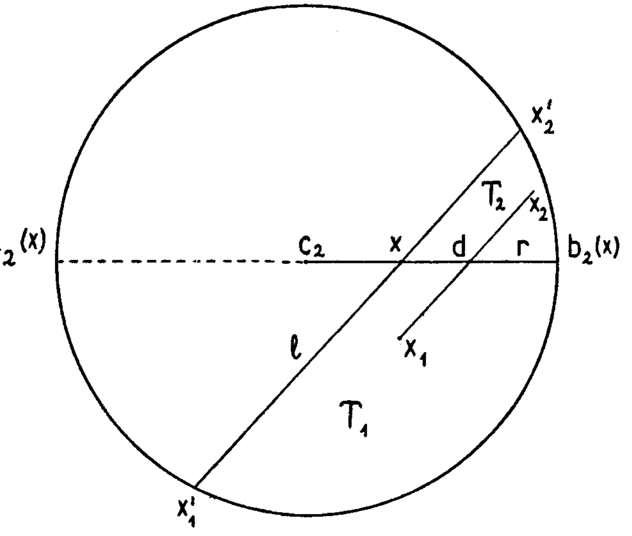

Figure 1

(2) $$\|x - x_i\| < \|x - x_i'\|.$$

Indeed, let T_i, $i = 1, 2$, be the triangle bounded by the segment $x\ x_i'$ (on l), by the segment $x\ b_2(x)$ (on r) and by the smaller arc of ∂S_2 between $b_2(x)$ and x_i'. As $T_1 \cup T_2$ is the segment of S_2 cut off by l which does not contain the center c_2, it follows that the angle of T_i at x_i' is smaller than $\pi/2$. T_i is thus contained in the disk with center x and radius $\|x - x_i'\|$. As $x_i \in T_i$, this implies (2). We choose again $x_3' = \cdots = x_m' = a_2(x)$ and thus proved the strong version for $k = 2$.

We now assume that it holds for all H with dim $H < k$ and prove it for dim $H = k (k \geqq 3)$. We have again three cases. (a′) $x = c_k$. As in (a), we put the x_i' in the endpoints of an arbitrary diameter.

(b′) $x \in H_l$, where $H_l = H(x_1, \cdots, x_p)$ is of dimension l with $1 \leqq l < k$ (hence $l < p < m$). Let P_l be the l-flat carrying H_l and $S_l = S_k \cap P_l$. By the assumption of our induction we have p points y_i in the solid l-sphere S_l, $y_1 \neq y_2$, $y_1 \neq a_l(x)$ and $y_3 = \cdots = y_p = a_l(x)$, such that x lies on the segment $y_1 y_2$ and such that

$$(3) \qquad \prod_{i=1}^{p} \| x - x_i \| \leqq \prod_{i=1}^{p} \| x - y_i \|.$$

We set $x_1' = y_1$, $x_2' = y_2$ and put $x_3' = \cdots = x_p' = x_{p+1}' = \cdots = x_m' = a_k(x)$. (Only if $c_k \in P_l$, then $a_l(x) = a_k(x)$; but always $x_1'(= y_1) \neq a_k(x)$.) (3) and the definition of $a_k(x)$ imply (1) and the strong version is established for this case (b′). Note that this covers the case $x \in \partial H$.

(c′) $x \in \operatorname{int} H$, $x \neq c_k$. Let again r be the radius through x ($r = c_k b_k(x)$) and let d be its intersection with ∂H. If d is a vertex of H or, more general, if d is one of the points x_i, then we set $x_1' = d$ and $x_2' = x_3' = \cdots = x_m' = a_k(x)$. If $d \neq x_i$, $i = 1, \cdots, m$, then it is an interior point of a l-dimensional face H_l of the convex polytope $H (H_l \subset \partial H)$, $1 \leqq l < k$. Let $x_1, \cdots, x_p$ $(l < p < m)$ be those points of the original set $(x_1, \cdots, x_m)$ which lie in H_l. Then $H_l = H(x_1, \cdots, x_p)$. Let Q_l be the l-flat carrying H_l. As d is an interior point of H_l, it follows that d is the only point on the radius r and in Q_l: $c_k \notin Q_l$, $x \notin Q_l$. Let P_l be the l-flat through x parallel to Q_l: $c_k \notin P_l$. For $i = 1, \cdots, p$ let r_i be the radius of S_k going through x_i and denote $P_l \cap r_i = z_i$. We thus project $H_l = H(x_1, \cdots, x_p)$, $H_l \subset Q_l$, from c_k into the l-polytope $H_l' = H(z_1, \cdots, z_p)$, $H_l' \subset P_l$. $d \in \operatorname{int} H_l$ implies $x \in \operatorname{int} H_l'$. We project once more. This time the p points z_i are projected from x onto ∂S_k; i.e. let l_i^* be the ray from x through z_i ($l_i^* \subset P_l$) and denote $\partial S_k \cap l_i^* = x_i''$.

Clearly, $H_l'' = H(x_1'', \cdots, x_p'')$ is a convex l-polytope, $H_l' \subset H_l''$ and $x \in \operatorname{int} H_l''$. (Figure 2 illustrates our construction for $k = 3$, $l = 2$ and $p = 3$. The parallel triangles H_2 and H_2' are not necessarily normal to r. H_2' and H_2'' lie in the same plane P_2 but are in general not similar.)

For each i, $i = 1, \cdots, p$, let P^i be the plane (2-flat) defined by the radii r and r_i. c_k, x and d are on r; c_k, z_i and x_i are on r_i; d and x_i are in Q_l and therefore on the line $P^i \cap Q_l$; x, z_i and x_i'' are in P_l and therefore on the parallel line $l_i = P^i \cap P_l$. (l_i contains the ray l_i^* from x through z_i.) All these points lie in the disk $S_2^i = S_k \cap P^i$ with the center $c_2^i = c_k$. c_2^i, x and d lie in this order on the radius r of S_2^i; $x_i \notin r$ and x_i'' is the intersection point of l_i^* with ∂S_2^i, where l_i^* is the ray from x parallel to the segment $d x_i$. x_i and x_i'' are on the same side of r. We thus have the same situation as in case (c) of $k = 2$. (We use now x_i'' instead of x_i'.) In analogy to (2), it follows that

$$(4) \qquad \| x - x_i \| < \| x - x_i'' \|, \qquad i = 1, \cdots, p.$$

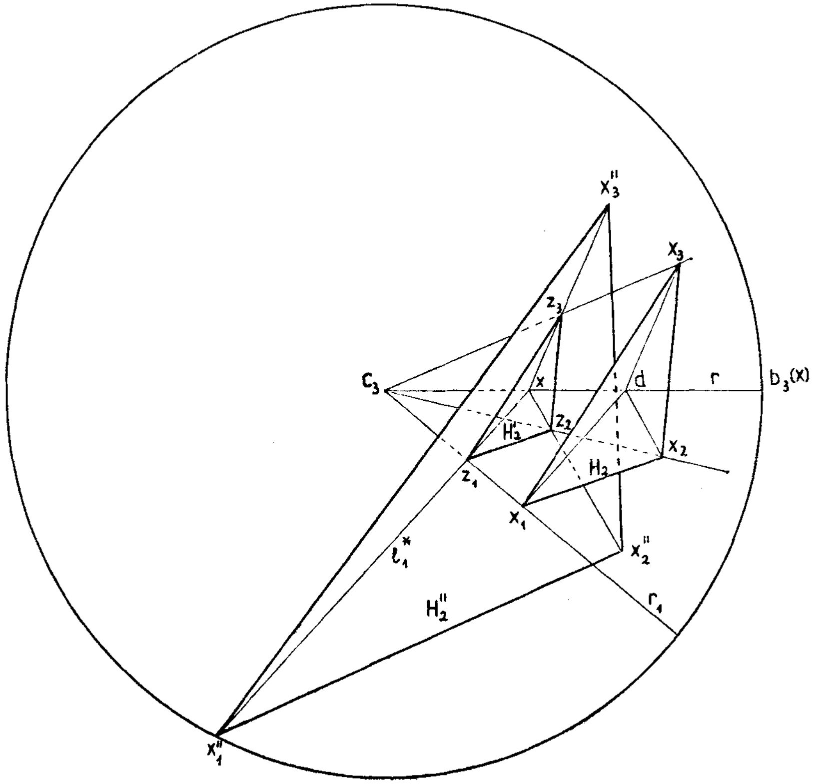

Figure 2

(In Figure 2 we assumed that P^1 is the plane of drawing. Compare with the lower half of Figure 1.)

Consider now the m points $x_1'', \cdots, x_p'', x_{p+1}, \cdots, x_m$. They all belong to S_k, x lies in their convex hull and, by (4), the product of the distances of x from these points is larger than the original product $\prod_{i=1}^m \| x - x_i \|$. If, by chance,

$$\dim H(x_1'', \cdots, x_p'', x_{p+1}, \cdots, x_m) < k$$

(i.e. if a vertex of the original $H = H(x_1, \cdots, x_m)$ lies in P_l), then the strong version follows by the assumption of our induction. If not, then we only note that now p of the points, namely $x_1'', \cdots, x_p''$, belong to the l-flat P_l $(1 \leqq l < k)$ and that $x \in H(x_1'', \cdots, x_p'')$ $(= H_l'')$. We therefore reduced this case (c′) to the former case (b′) and thus completed the proof of the strong version. Theorem 1 is thus established.

This theorem implies our main result, which we formulate only for the solid unit sphere $U_n = \{x: \| x \| \leqq 1\}$ $(c_n = 0)$ of E_n.

THEOREM 2. *Let* $x, x_1, \cdots, x_m (m \geqq 2)$ *be (not necessarily distinct) points of the solid unit sphere* U_n *of* E_n *such that* x *belongs to the convex hull of* $x_1, \cdots, x_m$. *Then*

$$\prod_{i=1}^{m} \|x - x_i\| \leqq (1 - \|x\|)\,(1 + \|x\|)^{m-1}. \tag{5}$$

For $0 < \|x\| < 1$ *equality holds in* (5) *only under the following conditions:* $\|x_i\| = 1$, $i = 1,\cdots,m$, $m-2$ *of the* x_i *coincide with the point* $a(x)$ *of* U_n *farthest away from* x, *and* x *lies on the chord bounded by the two remaining points.*

Proof. If x coincides with one of the x_i, then there is nothing to prove; this must happen if $\|x\| = 1$. If $x = 0$, then (5) is obvious and equality holds only if $\|x_i\| = 1$, $i = 1,\cdots,m$. Let $0 < \|x\| < 1$, $x \neq x_i$, $i = 1,\cdots,m$. By Theorem 1 we can assume that x is an interior point of the edge x_1x_2 of $H = H(x_1,\cdots,x_m)$. If we do not already have the situation mentioned in the equality statement, then we move x_1 and x_2 into the endpoints of their chord and, for $m \geqq 3$, move $x_3,\cdots,x_m$ into $a(x)$. This increases the product and yields the conditions of the equality statement. But now $\prod_{i=3}^{m}\|x - x_i\| = \|x - a(x)\|^{m-2} = (1 + \|x\|)^{m-2}$. The product $\|x - x_1\|\,\|x - x_2\|$ of the segments of a chord through x depends only on x. Hence, denoting the point of ∂U_n nearest to x by $b(x)$, we have

$$\|x - x_1\|\,\|x - x_2\| = \|x - b(x)\|\,\|x - a(x)\| = (1 - \|x\|)\,(1 + \|x\|).$$

This completes the proof of Theorem 2 and shows also that the strong version of Theorem 1 can be further strengthened: $x_1' = b_k(x)$, $x_2' = x_3' = \cdots = x_m' = a_k(x)$.

The function $(1 - \|x\|)\,(1 + \|x\|)^{m-1}$ attains its maximum only at $\|x\| = (m-2)/m$. We thus obtain from Theorem 2 the following

COROLLARY. *The assumptions of Theorem 2 imply*

$$\prod_{i=1}^{m} \|x - x_i\| \leqq \left(\frac{2}{m}\right)^{m}(m-1)^{m-1}. \tag{6}$$

Equality holds in (6) *only under the following conditions:* $\|x\| = (m-2)/m$, $\|x_i\| = 1$, $i = 1,\cdots,m$, $m-2$ *of the* x_i *coincide with* $a(x)$, *and* x *lies on the chord bounded by the two remaining points.*

2. **Szegö's maximum principle.** We show that the above corollary is also a consequence of the maximum principle for the product $\prod_{i=1}^{m}\|x - x_i\|$. (See Pólya–Szegö [4, section III problem 301]; and [5], [3].) The proof in [4, p. 328] not merely shows that this product attains its maximum in any bounded closed region D of E_n only at ∂D, but establishes that for any plane P the maximum in $P \cap D$ is taken only at $\partial(P \cap D)$. For convex polytopes this implies

THEOREM 3. *Let* $x_1,\cdots,x_m$ *be points in* E_n *such that at least two of them are distinct. For* x *varying in the convex hull* $H = H(x_1,\cdots,x_m)$, $\prod_{i=1}^{m}\|x - x_i\|$ *attains its maximum only at interior points of edges of* H.

For completeness we outline the proof. For dim $H = 1$ there is nothing to prove. For dim $H = 2$ identify the plane carrying H with the plane of the complex numbers and apply the (ordinary) maximum principle to the polynomial $\prod_{i=1}^{m}(z - z_i)$. For dim $H = k$, $2 < k \leqq n$, it suffices to consider the k-flat E_k carrying H. Let $\varepsilon > 0$ and set $H_\varepsilon = H - \bigcup_{i=1}^{m} \{\|x - x_i\| < \varepsilon\}$; for small ε the maximum of the product will be taken in H_ε. Let P be any plane such that $P \cap H_\varepsilon \neq \emptyset$. Choose coordinates $(\xi_1, \cdots, \xi_k)$ in E_k such that P satisfies $\xi_i = c_i$, $i = 3, \cdots, k$. For $x \in P$ set $\log \prod_{i=1}^{m} \|x - x_i\| = f(\xi_1, \xi_2)$. If $x \in P \cap H_\varepsilon$, then $f(\xi_1, \xi_2) \in C^2$. Using that not all the x_i lie in P, we obtain $\partial^2 f/\partial \xi_1^2 + \partial^2 f/\partial \xi_2^2 > 0$. This excludes the possibility of a maximum at interior points of $P \cap H_\varepsilon$. Varying the c_i $(i = 3, \cdots, k)$, it follows that $\prod_{i=1}^{m} \|x - x_i\|$ takes its maximum only at ∂H. If this maximum were taken at an interior point of a l-dimensional face of H with $l > 1$, then we would again obtain a contradiction by intersecting (for $l > 2$) this face with planes or by considering (for $l = 2$) the plane carrying this face.

Theorem 3 implies the above corollary. Indeed, to find $\max \prod_{i=1}^{m} \|x - x_i\|$ for all sets $(x, x_1, \cdots, x_m)$ satisfying $\|x_i\| \leqq 1$ and $x \in H(x_1, \cdots, x_m)$, it suffices by Theorem 3 to consider only those sets for which x is an interior point of an edge of H. We continue as in the proof of Theorem 2 and show that for such a set the product is bounded by $(1 - \|x\|)(1 + \|x\|)^{m-1}$. Varying $\|x\|$, we obtain (6).

Theorem 2 itself does not follow from Theorem 3 and seems to require a geometric proof. The property stated in Theorem 1 for the solid spheres is not valid for all convex bodies C of E_n. Indeed, let C be a regular simplex with center c and let x_i, $i = 1, \cdots, n + 1$, be the vertices of C. As for any $y \in C$, $y \neq x_i$, $i = 1, \cdots, n + 1$, we have $\|c - y\| < \|c - x_i\|$, it follows that for any set of $n + 1$ points x_i' of C, such that c belongs to an edge of their convex hull H', the relation

$$\prod_{i=1}^{n+1} \|c - x_i'\| < \prod_{i=1}^{n+1} \|c - x_i\|$$

holds.

3. **Normed linear spaces.** Let $x_1, \cdots, x_m$ be points in the solid unit sphere of n-dimensional unitary (complex Euclidean) space. As this space is just the real Euclidean space of dimension $2n$, it follows that Theorem 2 and its Corollary hold also for this space. Moreover, as they deal only with the convex hull of m points, it follows that (5) and (6) and the corresponding equality statements remain valid for (real or complex) Hilbert space. For normed linear spaces the following result, related to the Corollary, holds.

THEOREM 4. *Let N be a (real or complex) normed linear space and let $U = \{x; \|x\| \leqq 1\}$ be its solid unit sphere. For $m \geqq 2$ set*

$$h_m(N) = \sup \prod_{=1}^{m} \|x - x_i\|, \tag{7}$$

where the supremum is taken over all sets $(x, x_1, \cdots, x_m)$ satisfying $x_i \in U$, $i = 1, \cdots, m$ and $x \in H(x_1, \cdots, x_m)$. Then

$$(8)\qquad \left(\frac{2}{m}\right)^m (m-1)^{m-1} \leqq h_m(N) \leqq \left(\frac{2}{m}\right)^m (m-1)^m.$$

Moreover,

$$(9)\qquad h_m(N) = \left(\frac{2}{m}\right)^m (m-1)^m$$

for spaces with the supremum norm ($l^\infty(n)$, $n \geqq m$; l^∞; L^∞) and for spaces with the L_1 norm ($l^1(n)$, $n \geqq m$; l^1; L^1).

The Corollary implies that in the Euclidean case ($l^2(n) = E_n$, any n; l^2; L^2) $h_m(N)$ attains the lower bound of (8).

Proof. To obtain the first inequality sign of (8) it suffices to consider any diameter of U: let $\| a \| = 1$ and set $x_1 = -a$, $x_2 = \cdots = x_m = a$ and $x = ((2-m)/m)\, a$.

To prove the second inequality of (8) we note that the assumptions on $(x, x_1, \cdots, x_m)$ are:

$$(10)\qquad \| x_i \| \leqq 1,\ x = \sum_{i=1}^m t_i x_i, \quad t_i \geqq 0, \quad \sum_{i=1}^m t_i = 1, \quad i = 1, \cdots, m.$$

This implies

$$(11)\qquad \begin{aligned} \| x - x_i \| &= \| (t_i - 1)x_i + \sum_{j \neq i} t_j x_j \| \leqq (1 - t_i) \| x_i \| + \sum_{j \neq i} t_j \| x_j \| \\ &\leqq (1 - t_i) + \sum_{j \neq i} t_j = 2(1 - t_i), \qquad i = 1, \cdots, m. \end{aligned}$$

Hence,

$$(12)\qquad \prod_{i=1}^m \| x - x_i \| \leqq 2^m \prod_{i=1}^m (1 - t_i) \leqq 2^m \left(1 - \frac{1}{m}\right)^m.$$

(8) is thus established.

If, under the assumption (10),

$$(13)\qquad \prod_{i=1}^m \| x - x_i \| = \left(\frac{2}{m}\right)^m (m-1)^m,$$

then, using (11) and (12), it follows that

$$(14)\qquad \| x_i \| = 1, \qquad t_i = \frac{1}{m}, \qquad i = 1, \cdots, m.$$

To prove (9) we give sets $(x, x_1, \cdots, x_m)$ satisfying (14) and (13). For $l^\infty(n)$ and $l^1(n)$, $n \geqq m$, we denote $x_i = (\xi_{i1}, \cdots, \xi_{in})$, $i = 1, \cdots, m$. For $l^\infty(n)$ we choose $\xi_{ik} = 1 - 2\delta_{ik}$ $(i = 1, \cdots, m;\ k = 1, \cdots, n)$. Then

$$\| x - x_i \|_\infty = \max_k \left| \left(\frac{1}{m} - 1\right)\xi_{ik} + \frac{1}{m} \sum_{j \neq i} \xi_{jk} \right| = \frac{2m-2}{m}.$$

This proves (13) and the same example holds for l^∞. For $L^\infty(0,1)$ we may e.g. take m distinct Rademacher functions.

For $l^1(n)$ we choose $\xi_{ik} = \delta_{ik}$ $(i = 1,\cdots,m;\ k = 1,\cdots,n)$. Then

$$\| x - x_i \|_1 = \sum_{k=1}^{m} \left| \frac{1}{m} - \delta_{ik} \right| = \frac{2m-2}{m}.$$

The same example holds for l^1 and for $L^1(0,1)$ we may e.g. take $x_i(t) = m\delta_{ik}$ for $(k-1)/m < t < k/m$ $(i,k = 1,\cdots,m)$. This completes the proof of Theorem 4. Note that in the just considered cases the dimension of the maximizing $H(x_1,\cdots,x_m)$ is necessarily $m-1$; this is in contrast with the Euclidean case.

We add the following remark. If N is finite dimensional, then, owing to the compactness of the solid unit sphere, the supremum in (7) has to be attained. If (9) holds for a finite dimensional space, it follows therefore that there exist m points x_i, $\|x_i\| = 1$ satisfying

$$(15)\quad \left\| \left(\frac{1}{m} - 1\right) x_i + \frac{1}{m} \sum_{j \neq i} x_j \right\| = \left(1 - \frac{1}{m}\right) \| x_i \| + \frac{1}{m} \sum_{j \neq i} \| x_j \|;\ i = 1,\cdots,m.$$

As for $1 < p < \infty$ equality in Minkowski's inequality implies linear dependence, (15) cannot hold for $m > 2$ and $N = l^p(n)$ with $1 < p < \infty$. More general, by a result of Achieser and Krein (quoted in [1, p. 82] and [2, p. 112]) it follows that (15) cannot hold for rotund spaces (if $m > 2$): for finite dimensional rotund spaces and $m > 2$ the second inequality sign of (8) is strict.

4. **The absolute value of a polynomial at critical points.** We conclude with a simple application of Theorem 2.

THEOREM 5. *Let* $p_m(z) = \prod_{i=1}^{m} (z - z_i)$, $m \geqq 2$, *and assume that* $|z_i| \leqq 1$, $i = 1,\cdots,m$. *Then* $p_m'(z_0) = 0$ *implies*

$$(16)\qquad |p_m(z_0)| \leqq (1 - |z_0|)(1 + |z_0|)^{m-1}.$$

For $m \geqq 3$ *and* $z_0 \neq 0$, $|z_0| \neq 1$ *equality holds in* (16) *only if*

$$z_0 = \frac{m-2}{m} e^{i\alpha} \text{ and } p_m(z) = (z - e^{i\alpha})(z + e^{i\alpha})^{m-1}.$$

Proof. By the theorem of Gauss and Lucas [4, section III, problem 31] $p_m'(z_0) = 0$ implies $z_0 \in H(z_1,\cdots,z_m)$. Inequality (5) of Theorem 2 (for E_2) gives (16).

Let now $m \geqq 3$ and $0 < |z_0| < 1$. By Theorem 2 we can have equality in (16) only if $|z_i| = 1$ for all i and if $m-2$ of these roots coincide, say

$$z_3 = \cdots = z_m = -e^{i\alpha}.$$

If $\dim H = 2$, then it follows from the Gauss-Lucas theorem that the two critical points different from $-e^{i\alpha}$ are in the interior of the triangle H and Theorem 2 excludes equality in (16). There remains thus only the one dimensional case:

$z_1 = e^{i\alpha}$, $z_2 = z_3 = \cdots = z_m = -e^{i\alpha}$; the only critical point different from $-e^{i\alpha}$ is $z_0 = ((m-2)/m)e^{i\alpha}$ and in this case equality holds in (16).

For $m = 2$ equality holds in (16) if $|z_1| = |z_2| = 1$ $(z_0 = (z_1 + z_2)/2)$. For $m \geqq 3$ the other cases of equality are trivial: If $p_m'(0) = 0$ and all $|z_i| = 1$, then both sides of (16) equal 1, and at multiple roots on the unit circle both sides vanish.

References

1. D.F. Cudia, *Rotundity*, American Mathematical Society, Symposium on Convexity, eattle, Proc. Symp. Pure Math. 7 (1963).
2. M.M. Day, Normed linear spaces (2nd printing), Springer Verlag, Berlin, (1962).
3. Obreschkoff, N., Lösung der Aufgabe No. 10, *Jber, Deutsch. Math.-Verein.*, **33** (1925), 30.
4. G. Pólya and G. Szegö Aufgaben und Lehrsätze aus der Analysis, **1**, Dover, New York (1945).
5. G. Szegö, Aufgabe No. 10, *Jber. Deutsch. Math.-Verein*, **32** (1923), 16.

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